Week 11 Integration of rational functions

Recall: By long division and partial fractions

Rational function = polynomial + partial fractions

(Integer analogue : $\frac{9}{4} = 2 + \frac{1}{4}$)

eg
$$\frac{4x^6+x^4-1}{x^4-1} = 4x^2+1+\frac{4x^2}{x^4-1}$$
$$= 4x^2+1+\frac{1}{x-1}-\frac{1}{x+1}+\frac{2}{x^2+1}$$

$$\Rightarrow \int \frac{4x^6+x^4-1}{x^4-1}dx = \frac{4}{3}x^3+x+\ln|x-1| - \ln|x+1| + 2\arctan x + C$$

Integrate polynomial: Easy

Integrate partial fractions : Standard

Terms appear in partial fraction:

- $\frac{A}{ax+b}$ or $\frac{A}{(ax+b)^k}$, $k>1$: Easy to integrate
  Let $u = ax+b$

- $\frac{Ax+B}{ax^2+bx+c}$ or $\frac{Ax+B}{(ax^2+bx+c)^k}$, $k>1$, ax^2+bx+c is irreducible
  $\Delta = b^2-4ac<0$

Also use reduction formula for integration

Useful formula

$a>0$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$$

$$\int \tan\theta \, d\theta = \ln|\sec\theta| + C$$

Prove them !

eg
$$\int \frac{3}{2x+1}dx = \frac{3}{2}\int \frac{d(2x+1)}{2x+1} = \frac{3}{2}\ln|2x+1|+C$$

eg
$$\int \frac{3}{(2x+1)^4}dx = \frac{3}{2}\int \frac{d(2x+1)}{(2x+1)^4} = \frac{3}{2}\left(-\frac{1}{3}\right)\frac{1}{(2x+1)^3}+C = -\frac{1}{2(2x+1)^3}+C$$

eg $\displaystyle\int \frac{4x+7}{x^2+2x+5}\,dx$

Method I

$d(x^2+2x+5)=(2x+2)\,dx$ ①

$4x+7=2(2x+2)+3$ ②

$$\int \frac{4x+7}{x^2+2x+5}\,dx$$

$$\overset{②}{=} \int \frac{2(2x+2)}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx$$

$$\overset{①}{=} 2\int \frac{d(x^2+2x+5)}{x^2+2x+5} + \int \frac{3\,d(x+1)}{(x+1)^2+2^2}\,dx$$

$$= 2\ln|x^2+2x+5| + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C$$

Method II

$x^2+2x+5=(x+1)^2+2^2$

Let $x+1=2\tan\theta$, $x=2\tan\theta-1$

$dx=2\sec^2\theta\,d\theta$

$$\int\frac{4x+7}{x^2+2x+5}\,dx=\int\frac{8\tan\theta-4+7}{(2\tan\theta)^2+2^2}\,2\sec^2\theta\,d\theta$$

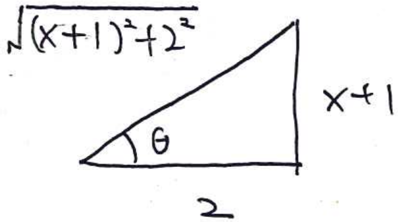
$$=\frac{8\tan\theta+3}{4\sec^2\theta}\cdot 2\sec^2\theta\,d\theta$$

$$=\frac{1}{2}\int(8\tan\theta+3)\,d\theta$$

$$=4\ln|\sec\theta|+\frac{3}{2}\theta+C$$

$$=4\ln\left|\frac{\sqrt{x^2+2x+5}}{2}\right|+\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

Are they both correct?



Rmk $4\ln\left|\frac{\sqrt{x^2+2x+5}}{2}\right| = 2\ln\left(\frac{x^2+2x+5}{4}\right) = 2\ln(x^2+2x+5)-2\ln 4$

$= 2\ln|x^2+2x+5|-2\ln 4$

$(\because\ x^2+2x+5>0\ \forall x)$

eg $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sec^4\theta}\sec^2\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

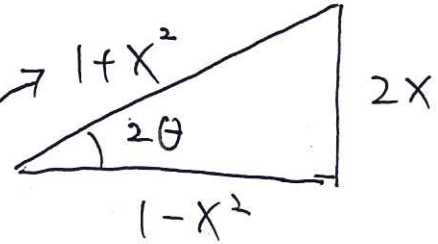
$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

$$= \frac{1}{2}\arctan x + \frac{1}{4}\frac{2x}{1+x^2} + C$$

$\circledast$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2x}{1-x^2}$$



$$\boxed{\begin{aligned}&(2x)^2+(1-x^2)^2\\ &= 4x^2+1-2x^2+x^4\\ &= 1+2x^2+x^4\\ &= (1+x^2)^2\end{aligned}}$$

$$\therefore \sin 2\theta = \frac{2x}{1+x^2}$$

Ex $\displaystyle\int \frac{3\,dx}{x^3+1} = ?$

Ans $= \ln|x+1| - \frac{1}{2}\ln(x^2-x+1) + \sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$

Summary of integrals of standard trig functions

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = -\ln|\cot x + \csc x| + C$$

$$= \ln\left|\tan\frac{t}{2}\right| + C \quad (t\text{-formula})$$

Pf $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{d\cos x}{\cos x}$

$(\cos x)^{-1} = \sec x \qquad = -\ln|\cos x| = \ln|\sec x|$

$\int \cot x\,dx$ is similar

$$\int \sec x\,dx$$

$$= \int \frac{\cos x}{\cos^2 x}\,dx$$

$$= \int \frac{d\sin x}{1-\sin^2 x}$$

$$= \frac{1}{2}\int\left(\frac{1}{\sin x+1} - \frac{1}{\sin x-1}\right) d\sin x$$

$$= \frac{1}{2}\left(\ln|\sin x+1| - \ln|\sin x-1|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\cdot\frac{\sin x+1}{\sin x+1}\right| + C$$

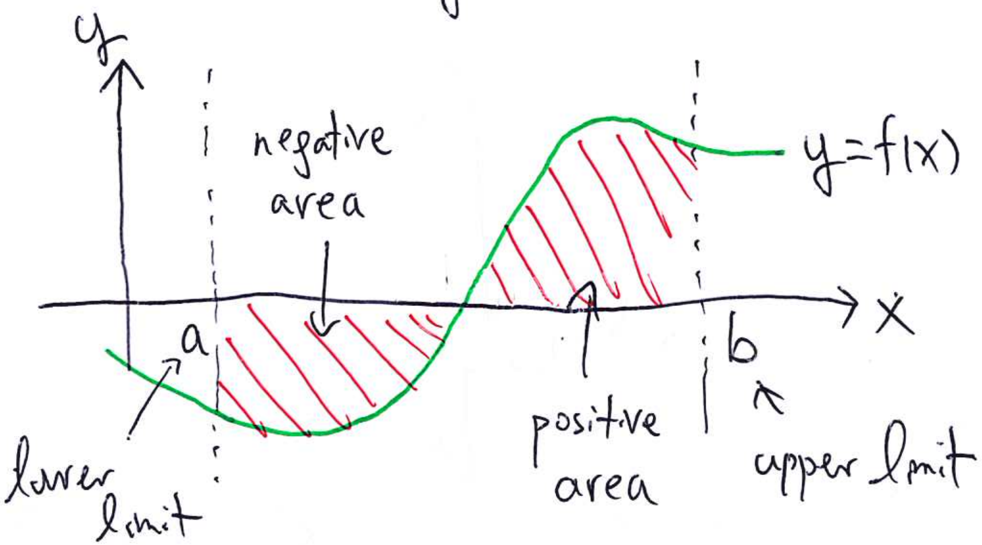
$$= \frac{1}{2}\ln\frac{(1+\sin x)^2}{\cos^2 x} + C$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| + C = \ln|\sec x + \tan x| + C$$

Definite Integral

Defn $a \leq b$

$$\int_a^b f(x)\,dx = \text{signed area under the graph of } y = f(x) \text{ between } x = a,\ x = b$$

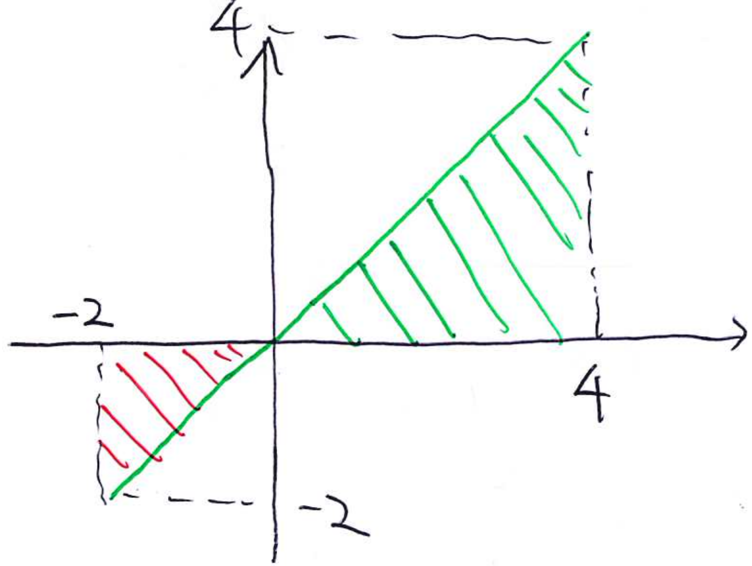


Rmk (1.) If $a > b$ $\quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

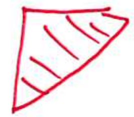
(2) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

x is a "dummy" variable

e.g. $f(x) = x$



$\int_2^4 x\,dx$ = Area of 

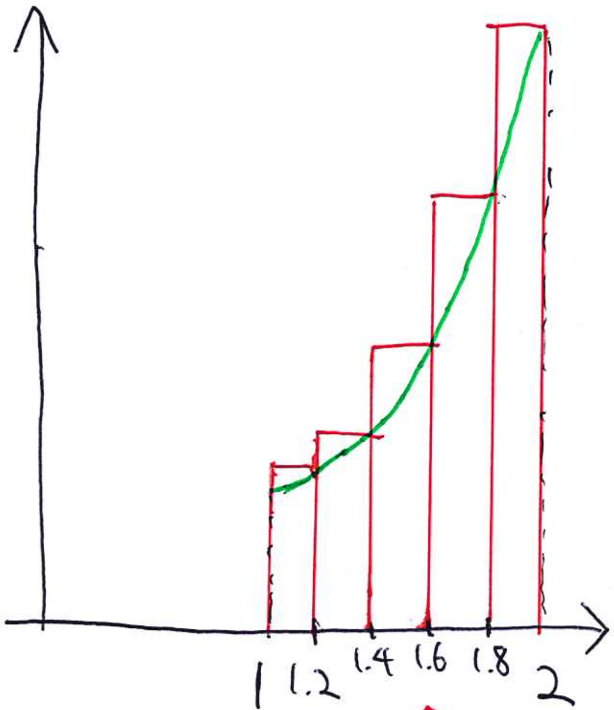
− area of 

$$= \frac{1}{2}(4)^2 - \frac{1}{2}(2)^2$$

$$= 8 - 2 = 6$$

Riemann Sum

Consider $f(x) = x^2$. $\int_1^2 x^2\,dx = ?$



Approximation:

eg $n=5$

Divide $[1,2]$ into n equal subintervals $I_1, I_2, \cdots, I_n$

length of each subinterval $= \dfrac{2-1}{n} = \dfrac{1}{n}$

$I_1 = [1, 1+\frac{1}{n}]$ $\quad I_2 = [1+\frac{1}{n}, 1+\frac{2}{n}]$

In general: $\quad I_k = [1+\frac{k-1}{n}, 1+\frac{k}{n}]$

Draw rectangles with base I_k and height $f(1+\frac{k}{n})$

Total area of rectangles
$$= \sum_{k=1}^{n} \frac{1}{n} f\left(1+\frac{k}{n}\right)$$
$$= \sum_{k=1}^{n} \frac{1}{n} \left(1+\frac{k}{n}\right)^2$$
$$= \sum_{k=1}^{n} \frac{1}{n} \left(1+\frac{2k}{n}+\frac{k^2}{n^2}\right)$$
$$= \frac{1}{n}\sum_{k=1}^{n} 1 + \frac{2}{n^2}\sum_{k=1}^{n} k + \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{1}{n}(n) + \frac{2}{n^2}\frac{1}{2}n(n+1) + \frac{1}{n^3}\frac{1}{6}n(n+1)(2n+1)$$

$$= 1 + \left(1+\frac{1}{n}\right) + \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Take limit $n \to \infty$

Area of rectangles $\longrightarrow$ Area under graph

$$\therefore \int_1^2 x^2\,dx = \lim_{n\to\infty} 1 + \left(1+\frac{1}{n}\right) + \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= 1 + (1+0) + \frac{1}{6}(1+0)(2+0)$$

$$= \frac{7}{3}$$

Rmk

Notation

$$\sum_{k=2}^{5} k^2 = 2^2 + 3^2 + 4^2 + 5^2$$

last term

first term

Formula

$$\sum_{k=m}^{n}\left(af(k) + bg(k)\right) = a\sum_{k=m}^{n} f(k) + b\sum_{k=m}^{n} g(k)$$

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$